

# Topology of Solutions of the Liouville Equation

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## Abstract

Suggestions concerning the generalization of the geometric quantization to the case of nonlinear field theories are given. Results for the Liouville field theory are presented.

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Significant part of cosmic rays is connected with the existence of black holes predicted by classical theory of gravitation. However, correct description of these data can be done only in the framework of quantum theory. Elementary particles produced in laboratories are well described by the standard model. This model, however, is a phenomenological one; it has nearly 20 free parameters.

To describe satisfactory both celestial and terrestrial data we should quantize, in a rigorous way, the Einstein general relativity and the Yang-Mills theory. Both theories are *nonlinear field* theories. The problem is that such theories have *infinitely* many degrees of freedom and the space of solutions to the field equations *is not* a vector space.

We have a few methods of quantization. Geometric quantization [1] is a method that has sound mathematical foundation and properly describes simple systems with *finitely* many degrees of freedom. It is not clear, however, if we can generalize this method to the case of nonlinear field theories.

Both general relativity and Yang-Mills theories are rather *complex* theories. It is perhaps reasonable to apply geometric quantization first to *simple* integrable systems to establish the framework. An example of such a system is the 2-dim Liouville theory [2]. The initial value problem for the Liouville equation reads:

$$\left(\partial_t^2 - \partial_x^2\right) \varphi(t, x) + \frac{m^2}{2} \exp \varphi(t, x) = 0, \quad m > 0 \quad (1)$$

$$\begin{cases} \varphi(0, x) &= \phi(x) \\ \varphi_t(0, x) &= \pi(x) \end{cases} \quad (2)$$

Equation (1) appears, e.g., in the context of the  $SU(2)$  Yang-Mills theory [3]. Fields which minimize locally the Yang-Mills finite action in 4-dim Euclidean space have to satisfy the equation  $F_{\mu\nu} = \tilde{F}_{\mu\nu}$  (where  $\tilde{F}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\lambda\gamma}F^{\lambda\gamma}$ ). Solutions to this equation invariant under 3-dim rotations combined with gauge transformations satisfy Eq.(1). Complete solution of the Cauchy problem for the Liouville equation was obtained in late 70-ties [4]. One can easily check [4] that Eq.(1) is satisfied by the function:

$$\varphi(t, x) := -\log \left[ \frac{m^2}{16} F^2(t, x) \right], \quad (3)$$

where

$$F(t, x) := \chi_1(x+t)\psi(x-t) + \chi_2(x+t)\psi_1(x-t), \quad (4)$$

and where  $\chi_k, \psi_k$  ( $k = 1, 2$ ) are any functions that fulfill two requirements:

$$\begin{cases} \chi_1 \chi_2' - \chi_2 \chi_1' &= 1 \\ \psi_1 \psi_2' - \psi_2 \psi_1' &= 1 \end{cases} \quad (5)$$

The solution (3) can be singular if there exists  $(t_0, x_0) \in R^2$  such that  $F(t_0, x_0) = 0$ . Applying the impicite mapping theorem to Eqs. (3-5) one can prove [5, 4] the lemma:

$$\left( \exists (t_0, x_0) \in R^2 : F(t_0, x_0) = 0 \right) \longrightarrow \left( \begin{array}{l} \exists R \ni t \rightarrow x(t) \in R \text{ such that} \\ x(t_0) = x_0 \text{ and } F(t, x(t)) = 0 \end{array} \right)$$

The domain of the mapping  $x(\cdot)$  is the entire set of real numbers.

Solutions that satisfy the initial value data (2) specify  $\psi_k$  and  $\chi_k$  ( see [4]):

$$\begin{cases} \psi_k'' = u \psi_k \\ \chi_k'' = w \chi_k \end{cases} \quad \text{for } k = 1, 2 \quad (6)$$

where

$$\begin{cases} u := \frac{1}{16} [(\phi' - \pi)^2 - 4(\phi' - \pi)' + m^2 \exp \phi] \\ w := \frac{1}{16} [(\phi' + \pi)^2 - 4(\phi' + \pi)' + m^2 \exp \phi] \end{cases} \quad (7)$$

Corollary resulting from the lemma is that

$$((\phi, \pi) \in C^\infty(R) \times C^\infty(R)) \implies (\varphi \in C^\infty(R^2, R)) \quad (8)$$

From now on, we only consider the smooth solutions.

The space of initial data  $\mathcal{F} := C^\infty(R) \times C^\infty(R)$  is a Fréchet space. However, the set of solutions

$$\mathcal{M} := \left\{ \varphi \in C^\infty(R^2, R) \mid \varphi \text{ satisfy Eqs.(1) and (2)} \right\} \subset C^\infty(R^2, R)$$

cannot be a Fréchet space, since it is not a vector space.

It turns out that  $\mathcal{F}$  and  $\mathcal{M}$  are *homeomorphic*. The proof is elementary but lengthy. Details are given in Ref. [5]. Consecutive steps of the proof are roughly the following:

1. One divides the mapping  $S : \mathcal{F} \rightarrow \mathcal{M}$  into a few mappings:

$$\mathcal{F} \ni (\phi, \pi) \xrightarrow{S_1} (u, w) \xrightarrow{S_2} (\psi, \chi) \xrightarrow{S_3} \dots \xrightarrow{S_n} \varphi \in \mathcal{M}$$

and proves that  $S_1, S_2, \dots, S_n$  are continuous. Then,  $S := S_n \cdot \dots \cdot S_2 \cdot S_1$  is continuous as a composition of continuous mappings.

2. The inverse mapping is continuous as it is defined by:

$$S^{-1} : \mathcal{M} \ni \varphi \implies S^{-1}(\varphi) := (\varphi(0, \cdot), \varphi_t(0, \cdot)) \in \mathcal{F}$$

3. Since we consider the solutions of the Cauchy problem we get

$$S \cdot S^{-1} = \mathbf{1}_{\mathcal{M}} \quad \text{and} \quad S^{-1} \cdot S = \mathbf{1}_{\mathcal{F}}$$

This completes the proof.

The geometric quantization procedure assumes that the phase space of a given classical system has the structure of a *manifold*. In the case of a simple system with  $n$ -degrees of freedom the phase space is a manifold modelled on  $R^{2n}$ . In the case of a field theory we would like to have an object that we could call a manifold modelled on a Fréchet space. The homeomorphism of  $\mathcal{F}$  and  $\mathcal{M}$  gives  $\mathcal{M}$  the structure of a topological manifold modelled on  $\mathcal{F}$ . There is only a single chart  $(\mathcal{M}, S^{-1})$ .

The starting point in the geometric quantization procedure of a mechanical system is to express the evolution of the system on the phase space in terms of *symplectic* geometry. Can one follow this procedure in the case of the Liouville field theory? We shall try to answer this question in the near future [6].

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## References

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